Efficient wPINN-Approximations to Entropy Solutions of Hyperbolic Conservation Laws

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Physics-Informed Neural Networks (PINNs) have recently emerged as a deep learning framework for approximating solutions to partial differential equations. They have been successfully applied to a variety of forward- and inverse problems [1]. Particularly for high-dimensional systems of PDEs, classical mesh-based methods are computationally infeasible, while PINNs provide a promising avenue due to being a naturally mesh-free method.

However, for systems of nonlinear hyperbolic conservation laws, standard PINNs fail at approximating discontinuous solutions [2]. An example of a system with discontinuous solutions is the compressible Euler equation. We provide some explicit computations to outline the precise reason standard PINNs fail at approximating discontinuous solutions of nonlinear hyperbolic conservation laws.

An approach to solving this kind of PDE is learning solutions in an appropriate weak sense, as outlined recently in [3] for scalar nonlinear hyperbolic conservation laws. This strategy has been termed "weak PINNs" (wPINNs). The basic idea is training adversarial neural networks to estimate weak (dual) norms of the PDE residual.

We present our modifications to the wPINN strategy, that reformulate the dual norm estimation, to make it more efficient for learning and give smaller errors [4]. Additionally, our modified approach generalizes naturally to systems.

We showcase numerical experiments that compare our modifications to the original approach for the inviscid Burgers equation and demonstrate that our approach can solve the compressible Euler equations by simulating Sod's shock tube example.

References

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